

# THE BEHAVIOUR OF CURVATURE FUNCTIONS AT CUSPS AND INFLECTION POINTS

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ABSTRACT. At a 3/2-cusp of a given plane curve  $\gamma(t)$ , both of the Euclidean curvature  $\kappa_g$  and the affine curvature  $\kappa_A$  diverge. In this paper, we show that each of  $\sqrt{|s_g|}\kappa_g$  and  $(s_A)^2\kappa_A$  (called the *Euclidean* and *affine normalized curvature*, respectively) at a 3/2-cusp is a  $C^\infty$ -function of the variable  $t$ , where  $s_g$  (resp.  $s_A$ ) is the Euclidean (resp. affine) arclength parameter of the curve corresponding to the 3/2-cusp  $s_g = 0$  (resp.  $s_A = 0$ ). Moreover, we give a characterization of the behaviour of the curvature functions  $\kappa_g$  and  $\kappa_A$  at 3/2-cusps. On the other hand, inflection points are also singular points of curves in affine geometry. We give a similar characterization of affine curvature functions near generic inflection points. As an application, new affine invariants of 3/2-cusps and generic inflection points are given.

## 1. INTRODUCTION.

Let  $\gamma(t)$  be a smooth (i.e.  $C^\infty$ ) curve in the plane  $\mathbf{R}^2$  defined on an open interval containing  $t = 0$ . The origin  $t = 0$  is called a *singular point* of  $\gamma$  if  $\dot{\gamma}(t) = d\gamma(t)/dt$  vanishes at  $t = 0$ . Moreover, a singular point  $t = 0$  is called a *3/2-cusp* if there exist a suitable coordinate change  $t = t(s)$  and a local diffeomorphism  $\Phi$  of  $\mathbf{R}^2$  at  $\gamma(0)$  such that  $\Phi \circ \gamma \circ s(t) = {}^t(t^2, t^3)$ , where  ${}^t$  denotes the transpose operation on matrices. It is well-known that a singular point  $t = 0$  is a 3/2-cusp if and only if  $[\ddot{\gamma}(0), \gamma^{(3)}(0)] \neq 0$  holds, where  $[\mathbf{a}, \mathbf{b}]$  denotes the determinant, that is

$$[\mathbf{a}, \mathbf{b}] = a_1 b_2 - a_2 b_1, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Moreover, a 3/2-cusp  $t = 0$  is called a *positive cusp* (resp. a *negative cusp*) if  $[\ddot{\gamma}(0), \gamma^{(3)}(0)]$  is positive (resp. negative). This signature of cusps is invariant under an orientation preserving diffeomorphism of  $\mathbf{R}^2$ . If one reverses the orientation of the curve, the signature of the cusp changes. An invariant called the (Euclidean) *cuspidal curvature*  $\mu_g$  at 3/2-cusps is given as follows (which was introduced in [7], and its fundamental properties are given in [5])

$$(1.1) \quad \mu_g := \frac{[\ddot{\gamma}(0), \gamma^{(3)}(0)]}{|\ddot{\gamma}(0)|^{5/2}},$$

which is independent of orientation preserving isometries of  $\mathbf{R}^2$ . The sign of the cuspidal curvature coincides with that of cusps. Let  $\gamma_1(t)$  and  $\gamma_2(t)$  be two 3/2-cusps at  $t = 0$ . Suppose that  $\gamma_1$  and  $\gamma_2$  have the same *cuspidal curvature*, then

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there exists orientation preserving isometry  $T$  of  $\mathbf{R}^2$  and a parametrization  $u = u(t)$  near  $t = 0$  such that  $|T \circ \gamma_2 \circ u(t) - \gamma_1(t)|$  has order higher than  $t^3$ .

If we denote by  $s_g$  the (Euclidean) arclength parameter of the curve such that  $s_g = 0$  corresponds to a 3/2-cusp, then we show firstly in this paper that  $\sqrt{|s_g|} \kappa_g$  (called the ‘normalized curvature function’) is a  $C^\infty$ -function of the variable  $t$ , which induces the cuspidal curvature as follows:

**Theorem 1.1.** *Let  $\gamma(t) : (-\varepsilon, \varepsilon)$  ( $\varepsilon > 0$ ) be a 3/2-cusp in the Euclidean plane  $\mathbf{R}^2$  such that  $t = 0$  is a 3/2-cusp. Then  $\tau = \operatorname{sgn}(t)\sqrt{|s_g|}$  can be taken to be a coordinate of  $\gamma$  at  $t = 0$  (called the half-arclength parameter) and  $\sqrt{|s_g|} \kappa_g$  is a  $C^\infty$ -function of  $t$  (and  $\tau$ ). Moreover, it holds that*

$$(1.2) \quad \lim_{t \rightarrow 0} \sqrt{|s_g|} \kappa_g = \frac{\mu_g}{2\sqrt{2}},$$

where  $s_g = \int_0^t |\dot{\gamma}(u)| du$  is the (Euclidean) arclength parameter and  $\mu_g$  is the cuspidal curvature of  $\gamma$  at  $t = 0$  given in (1.1). Conversely, if we take a  $C^\infty$ -function  $f(\tau)$  such that  $f(0) \neq 0$ , then there exists a 3/2-cusp such that

$$(1.3) \quad \sqrt{|s_g|} \kappa_g = f(\tau),$$

and  $\tau$  is the half-arclength parameter.

The formula (1.2) was not given in [6] and [5]. For each smooth function  $\varphi(s)$ , it is well-known that there exists a regular curve  $\gamma(s)$  with arclength parameter whose curvature function is  $\varphi(s)$ . The last assertion of the theorem is an analogue of this fact for 3/2-cusps. Later, we also show that the same assertion holds for 3/2-cusps in an arbitrarily given Riemannian 2-manifold (cf. Theorem 2.6).

On the other hand, in affine geometry, Izumiya-Sano [3] pointed out the fact that the affine evolute having 3/2-cusps corresponding to sextactic points is exactly analogous to the fact that the Euclidean evolute has cusps corresponding to vertices. Related to this work, Giblin and Sapiro [2] studied the affine distance symmetry set from the viewpoint of singularity theory. These two works suggest that the 3/2-cusp is also an important object in affine geometry. In this paper, we define a new affine invariant for 3/2-cusps called the *affine cuspidal curvature*, by

$$(1.4) \quad \mu_A := \frac{24[\ddot{\gamma}, \gamma^{(3)}][\ddot{\gamma}, \gamma^{(5)}] + 60[\ddot{\gamma}, \gamma^{(3)}][\gamma^{(3)}, \gamma^{(4)}] - 35[\ddot{\gamma}, \gamma^{(4)}]^2}{[\ddot{\gamma}, \gamma^{(3)}]^{12/5}} \Big|_{t=0},$$

where  $t = 0$  is a 3/2-cusp of  $\gamma(t)$ . It is invariant under equi-affine transformations and independent of the choice of an orientation of the curve  $\gamma(t)$ . (An *equi-affine transformation* is an affine transformation whose Jacobian is  $\pm 1$  with respect to the canonical coordinate system of  $\mathbf{R}^2$ .) Let  $m, n$  be two mutually prime integers. Here (and also throughout in this paper), we use the convention for the fractional order of exponent as follows

$$(1.5) \quad t^{m/n} := (-1)^{mn} |t|^{m/n}.$$

For example,  $t^{1/2}$  is equal to  $\sqrt{|t|}$ . As an analogue of Euclidean cuspidal curvature, we show the following two assertions:

**Theorem 1.2.** *Let  $\gamma_1(t)$  and  $\gamma_2(t)$  be two 3/2-cusps in the affine plane  $\mathbf{R}^2$  at  $t = 0$ . Then  $\gamma_1$  and  $\gamma_2$  have the same affine cuspidal curvature if and only if there exists an equi-affine transformation  $T$  of  $\mathbf{R}^2$  and a parametrization  $u = u(t)$  near  $t = 0$  such that  $du/dt > 0$  and  $|T \circ \gamma_2 \circ u(t) - \gamma_1(t)|$  has order higher than  $t^5$ .*

**Theorem 1.3.** *Let  $\gamma(t) : (-\varepsilon, \varepsilon)$  ( $\varepsilon > 0$ ) be a 3/2-cusp in the affine plane  $\mathbf{R}^2$  such that  $t = 0$  is a 3/2-cusp and  $s_A$  is the affine arclength parameter (cf. (3.2)). Then  $\tau = (s_A)^{3/5}$  (cf. (1.5)) can be taken to be a coordinate of  $\gamma$  at  $t = 0$  (called the 3/5-arclength parameter), and  $f := (s_A)^2 \kappa_A$  is a  $C^\infty$ -function of  $t$  (and  $\tau$ ), called the normalized affine curvature function (see (3.1) for definition of  $\kappa_A$ ). Moreover, it satisfies  $f(0) = 4/25$ ,  $\dot{f}(0) = 0$  and*

$$(1.6) \quad \lim_{t \rightarrow 0} \frac{(s_A)^2 \kappa_A - 4/25}{\tau^2} = \frac{1}{220} \sqrt[5]{\frac{20}{3}} \mu_A.$$

*Conversely, if we take a  $C^\infty$ -function  $f(\tau)$  such that  $f(0) = 4/25$  and  $\dot{f}(0) = 0$ , then there exists a 3/2-cusp whose normalized affine curvature function is  $f(\tau)$  with respect to the 3/5-length parameter.*

Later, we generalize Theorem 1.3 for 3/2-cusps in an arbitrarily given 2-manifold with equi-affine structure (cf. Theorem 3.8).

A point  $t = 0$  on a regular curve  $\gamma(t)$  is called an *inflection point* if it satisfies  $[\dot{\gamma}(0), \ddot{\gamma}(0)] = 0$ . An inflection point is called *generic* if  $[\dot{\gamma}(0), \gamma^{(3)}(0)] \neq 0$  holds. Affine geometry of curves usually requires higher order derivatives than those in Euclidean geometry, and this fact is often connected to several interesting phenomena different from Euclidean geometry: For example, inflection points are singular points in affine geometry, as well as 3/2-cusps. In Section 3, we show an analogue of Theorems 1.2 and 1.3 for inflection points. In particular, the normalized (affine) curvature function  $f(t) = (s_A)^2 \kappa_A$  is a  $C^\infty$ -function at a generic inflection point  $t = 0$ , which satisfies  $f(0) = -5/16$  and also the following non-trivial identity (see Theorem 4.4 and Theorem 4.7)

$$(1.7) \quad -\frac{9}{7} \left( \frac{([\dot{\gamma}(0), \gamma^{(4)}(0)] + [\ddot{\gamma}(0), \gamma^{(3)}(0)])}{[\dot{\gamma}(0), \gamma^{(3)}(0)]} \right) \dot{f}(0) + 32\dot{f}(0)^2 + 9\ddot{f}(0) = 0.$$

This corresponds to the fact that  $\dot{f}(0)$  vanishes at 3/2-cusps, as in Theorem 1.3.

## 2. THE NATURAL EQUATION OF CUSPS IN EUCLIDEAN GEOMETRY

We denote by  $C_0^\infty(\mathbf{R})$  the set of germs of real-valued  $C^\infty$ -functions defined at  $t = 0$ . Let  $\gamma(t)$  be a curve defined on an interval  $(-\delta, \delta)$  for  $\delta > 0$ . We suppose that  $t = 0$  is a 3/2-cusp, namely it satisfies  $\dot{\gamma}(0) = \mathbf{0}$  and  $[\ddot{\gamma}(0), \gamma^{(3)}(0)] \neq 0$ . Then it can be easily checked that the Euclidean curvature function

$$(2.1) \quad \kappa_g(t) := \frac{[\dot{\gamma}(t), \ddot{\gamma}(t)]}{|\dot{\gamma}(t)|^3}$$

diverges. More precisely, the following assertion holds:

**Lemma 2.1.** *Let  $t = 0$  be a 3/2-cusp of the curve  $\gamma(t)$  in the Euclidean plane  $\mathbf{R}^2$ . Then,  $\tau := \operatorname{sgn}(t)\sqrt{|s_g|}$  can be taken as a local coordinate of the curve  $\gamma$  at  $t = 0$ . Moreover,  $\sqrt{|s_g|}\kappa_g$  belongs to  $C_0^\infty(\mathbf{R})$ , and (1.2) holds.*

*Proof.* Without loss of generality, we may assume that  $\gamma(0) = \mathbf{0}$ . We may set

$$(2.2) \quad \gamma(t) = \frac{\ddot{\gamma}(0)}{2}t^2 + \frac{\gamma^{(3)}(0)}{6}t^3 + t^4\Gamma(t),$$

where  $\Gamma(t)$  is a  $\mathbf{R}^2$ -valued  $C^\infty$ -function. Then we get the following expression

$$(2.3) \quad [\dot{\gamma}(t), \ddot{\gamma}(t)] = t^2 \left( \frac{[\ddot{\gamma}(0), \gamma^{(3)}(0)]}{2} + t\varphi_1(t) \right),$$

where  $\varphi_1(t) \in C_0^\infty(\mathbf{R})$ . Similarly, there exists  $\varphi_2(t) \in C_0^\infty(\mathbf{R})$  such that

$$(2.4) \quad |\dot{\gamma}(t)| = |t| |\ddot{\gamma}(0) + t\varphi_2(t)|.$$

Then (2.3) and (2.4) imply that  $|t|\kappa_g(t) \in C_0^\infty(\mathbf{R})$ . On the other hand, applying Lemma .9 below by setting  $\alpha = 1$  and  $\varphi(t) := |\ddot{\gamma}(0) + t\varphi_2(t)|$ , we can conclude that

$$f(t) := \frac{s_g(t)}{\text{sgn}(t)t^2} = \frac{|s_g(t)|}{t^2} = \left( \frac{\sqrt{|s_g(t)|}}{|t|} \right)^2$$

is a  $C^\infty$ -function such that  $f(0) > 0$ . Then

$$\tau := \text{sgn}(t)\sqrt{|s_g(t)|} = t\sqrt{f(t)}$$

is a  $C^\infty$ -function of  $t$ . Since  $d\tau(0)/dt = f(0) > 0$ , the function  $\tau$  can be taken as a local coordinate of the curve  $\gamma$  at  $t = 0$ . Since  $|t|\kappa_g \in C_0^\infty(\mathbf{R})$ , the function

$$(2.5) \quad \sqrt{|s_g|}\kappa_g = (|t|\kappa_g)\sqrt{f(t)}$$

is also a  $C^\infty$ -function of  $t$ . Finally, the formula (1.2) follows directly from (2.3), (2.4), (2.5) and  $f(0) = |\ddot{\gamma}(0)|/2$  (cf. (A.1) in the appendix).  $\square$

*Remark 2.2.* A parametrization  $t$  of the 3/2-cusp  $\gamma(t)$  in  $\mathbf{R}^2$  is the half-arclength parameter if and only if it satisfies  $|\dot{\gamma}| = 2|t|$ .

*Proof of Theorem 1.1.*

The first parts of the assertion have been proved in Lemma 2.1. So it is sufficient to show the last assertion. We take a  $C^\infty$ -function  $f(\tau)$ . Let  $\gamma(\tau)$  be a curve defined by

$$(2.6) \quad \gamma(\tau) := 2 \int_0^\tau u \begin{pmatrix} \cos \theta(u) \\ \sin \theta(u) \end{pmatrix} du \quad \left( \theta(\tau) := 2 \int_0^\tau f(u) du \right).$$

Then it holds that  $|\dot{\gamma}| = 2\tau$ , which implies that  $\tau$  is the half-arclength parameter. Moreover, one can directly check that  $\gamma(\tau)$  has a 3/2-cusp satisfying (1.3).  $\square$

*Example 2.3.* The arclength parameter of the cusp

$$\gamma(t) = a \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \quad (a > 0)$$

is given by

$$s_g = \text{sgn}(t)(\varphi(t) - 8)a \quad \left( \varphi(t) := (9t^2 + 4)^{3/2} \right),$$

which is a continuous function, but not smooth at  $t = 0$ . The half-arclength parameter  $\tau = \text{sgn}(t)a\sqrt{\varphi(t) - 8}$  is a smooth function at  $t = 0$  and the normalized curvature function is

$$\frac{2(\varphi(t) - 8)^{1/2}}{\sqrt{3at}\varphi(t)} = \frac{3}{4\sqrt{a}} - \frac{297t^2}{128\sqrt{a}} + o(t^3) \quad \left( \varphi(t) := (9t^2 + 4)^{3/2} \right),$$

which implies that the cuspidal curvature at  $t = 0$  is equal to  $3/\sqrt{2a}$ .

*Example 2.4.* The half-arclength parameter of the cycloid

$$(2.7) \quad a \begin{pmatrix} t - \sin t \\ -1 + \cos t \end{pmatrix} \quad (a > 0)$$

at  $t = 0$  is equal to  $2\sqrt{2a} \sin(t/4)$  and the normalized curvature function is given by

$$\frac{1}{2\sqrt{2a} \cos(t/4)} = \frac{1}{2\sqrt{2a}} + \frac{t^2}{64\sqrt{2a}} + o(t^3),$$

which implies that the cuspidal curvature at  $t = 0$  is equal to  $1/\sqrt{a}$ .

*Example 2.5.* The curve given by (cf. Figure 1)

$$\frac{1}{2a^2} \begin{pmatrix} 2a\tau \sin(2a\tau) + \cos(2a\tau) \\ \sin(2a\tau) - 2a\tau \cos(2a\tau) \end{pmatrix}$$

is called the *canonical 3/2-cusp*, which has the property that  $\sqrt{|s_g|}\kappa_g$  is identically equal to  $a$ , where  $\tau$  is the half-arclength parameter.

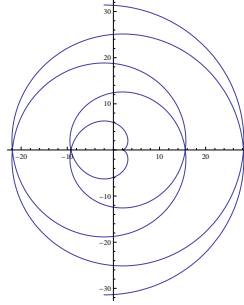


FIGURE 1. The canonical 3/2-cusp ( $a = 1$ ) in Euclidean geometry

To finish this section, we generalize Theorem 1.1 to 3/2-cusps in an arbitrary Riemannian 2-manifold: Let  $\gamma(t)$  ( $|t| < \delta$ ) be a regular curve in a given oriented Riemannian 2-manifold  $(M^2, g)$ . We denote by  $\Omega_g$  the unit area element of  $M^2$ . By definition,  $\Omega_g(\mathbf{e}_1, \mathbf{e}_2) = 1$  holds for a positively oriented orthonormal frame  $\mathbf{e}_1, \mathbf{e}_2$  of  $(M^2, g)$ . For the sake of simplicity, we set

$$[v, w] := \Omega_g(v, w) \quad (v, w \in T_p M^2, p \in M^2).$$

This notation fits with the previous one, since  $\Omega_g(v, w) = \det(v, w)$  holds on the Euclidean plane  $\mathbf{R}^2$ . Then the geodesic curvature  $\kappa_g$  of the curve is defined by exactly the same formula (2.1), where

$$\ddot{\gamma}(t) = \nabla_t \dot{\gamma}(t), \quad |\dot{\gamma}(t)| = \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}.$$

On the other hand, let  $\gamma(t)$  ( $|t| < \delta$ ) be a 3/2-cusp at  $t = 0$  in  $M^2$ . Then we define the *cuspidal curvature*  $\mu_g$  at  $t = 0$  by (1.1) like as in the case of  $\mathbf{R}^2$ , where

$$\gamma^{(3)} = \nabla_t \ddot{\gamma}, \quad \gamma^{(4)} = \nabla_t \gamma^{(3)}.$$

Then the following assertion holds, which is a generalization of Theorem 1.1.

**Theorem 2.6.** *Let  $\gamma(t) : (-\delta, \delta)$  ( $\delta > 0$ ) be a curve in an oriented Riemannian 2-manifold  $(M^2, g)$  such that  $t = 0$  is a 3/2-cusp. Then the same assertion as in Theorem 1.1 holds.*

To prove the assertion, we prepare the following assertion:

**Lemma 2.7.** *Let  $X(t)$  be a  $C^\infty$ -vector field along  $\gamma$  such that*

$$(2.8) \quad X(0) = \dot{X}(0) = \cdots = X^{(n)}(0) = \mathbf{0},$$

*where  $\dot{X} = \nabla_t X$  and  $X^{(i)} := \nabla_t X^{(i-1)}$  ( $i = 1, 2, 3, \dots$ ). Then there exist  $\delta > 0$  and a  $C^\infty$ -vector field  $Y(t)$  along  $\gamma(t)$  ( $0 \leq t < \delta$ ) such that  $X(t) = t^{n+1}Y(t)$ .*

*Proof.* Let  $(\mathbf{e}_1(t), \mathbf{e}_2(t))$  be a parallel frame field along  $\gamma$ . We may set

$$X(t) := f_1(t)\mathbf{e}_1(t) + f_2(t)\mathbf{e}_2(t),$$

and then it holds that

$$X^{(i)}(t) = f_1^{(i)}(t)\mathbf{e}_1(t) + f_2^{(i)}(t)\mathbf{e}_2(t) \quad (i = 1, 2, \dots, n).$$

Then (2.8) is equivalent to the condition  $f_j(0) = f_j^{(1)}(0) = \cdots = f_j^{(n)}(0) = 0$  ( $j = 1, 2$ ). Then, there exists a  $C^\infty$ -function  $g_j(t) \in C_0^\infty(\mathbf{R})$  such that  $f_j(t) = t^{n+1}g_j(t)$  ( $j = 1, 2$ ) and  $X = t^{n+1}(g_1\mathbf{e}_1 + g_2\mathbf{e}_2)$  holds.  $\square$

(*Proof of Theorem 2.6.*) The assertion of Lemma 2.1 holds for 3/2-cusps, however, the proof should be modified: Let  $v$  be a tangent vector of  $M^2$  at  $\gamma(0)$ . Then we denote by  $P_v(t)$  the parallel vector field along  $\gamma(t)$  such that  $P_v(0) = v$ . We set

$$(2.9) \quad X(t) := \dot{\gamma}(t) - P_2t - \frac{1}{2}P_3t^2,$$

where  $P_2 := P_{\dot{\gamma}(0)}(t)$  and  $P_3 := P_{\gamma^{(3)}(0)}(t)$ . Then it holds that

$$X(0) = \dot{X}(0) = \ddot{X}(0) = \mathbf{0}.$$

By Lemma 2.7, there exists a vector field  $\Gamma(t)$  along  $\gamma(t)$  such that

$$\dot{\gamma}(t) = P_2t + \frac{1}{2}P_3t^2 + t^3\Gamma(t).$$

Then it holds that

$$\ddot{\gamma}(t) = P_2 + P_3t + t^2(3\Gamma(t) + t\dot{\Gamma}(t)).$$

Using these expressions, one can easily prove the same assertion of Lemma 2.1 for a curve  $\gamma$  on the Riemannian manifold  $(M^2, g)$ . In particular,  $f(t) := \sqrt{|s_g|}\kappa_g$  is a smooth function of  $t$ , and  $\tau := \text{sgn}(t)\sqrt{|s_g|}$  gives a local coordinate of the curve at the 3/2-cusp. We can also check the identity (1.2) by the completely same argument as in the proof of Lemma 2.1.

However, our previous proof of the last assertion of Theorem 1.1 cannot be applied, since (2.6) holds only for the case that the ambient space is  $\mathbf{R}^2$ . To prove it here, we need the following new idea: Let  $\gamma(\tau)$  be a 3/2-cusp at  $\tau = 0$ , and suppose that  $\tau$  is the half-arclength parameter. We use the notations

$$(2.10) \quad \dot{\gamma} := \frac{d\gamma}{d\tau}, \quad \ddot{\gamma} := \frac{d^2\gamma}{d\tau^2}, \quad \gamma^{(3)} := \frac{d^3\gamma}{d\tau^3}, \quad \gamma^{(4)} := \frac{d^4\gamma}{d\tau^4}, \quad \dots,$$

and

$$(2.11) \quad \gamma' := \frac{d\gamma}{ds}, \quad \gamma'' := \frac{d^2\gamma}{ds^2}, \quad \gamma^{[3]} := \frac{d^3\gamma}{ds^3}, \quad \gamma^{[4]} := \frac{d^4\gamma}{ds^4}, \quad \dots,$$

where  $s = s_g$ . Since  $s_g = \tau^2$ , the following identities hold:

$$(2.12) \quad \dot{\gamma} = 2\tau\gamma', \quad \ddot{\gamma} = 2\mathbf{e} + 4\tau f(\tau)\mathbf{n},$$

where  $\mathbf{e} = \gamma'$  and  $\mathbf{n}$  is the unit normal vector field along  $\gamma$  such that  $[\mathbf{e}, \mathbf{n}] = 1$ . Here, we used the following identities

$$(2.13) \quad \dot{\mathbf{e}} = 2\tau\mathbf{e} = 2\tau\kappa_g\mathbf{n} = 2f(\tau)\mathbf{n}, \quad \dot{\mathbf{n}} = 2\tau\mathbf{n} = -2f(\tau)\mathbf{n}.$$

Using (2.12), one can reprove the identity (1.2), since both sides of (1.2) are independent of the choice of the parameter  $t$ . On the other hand, we have that

$$\gamma^{(3)} = -8\tau f^2\mathbf{e} + 4(\tau\dot{f} + 2f)\mathbf{n}.$$

Since  $\ddot{\gamma}$  and  $\gamma^{(3)}$  at  $\tau = 0$  are linearly independent, using the Schmidt orthonormalization of the frame  $\{\ddot{\gamma}(\tau), \gamma^{(3)}(\tau)\}$ , we get the following  $C^\infty$ -orthonormal frame field along  $\gamma$

$$\mathbf{u}_1 := \frac{1}{\sqrt{1 + 4\tau^2 f^2}}(\mathbf{e} + 2\tau f\mathbf{n}), \quad \mathbf{u}_2 := \frac{1}{\sqrt{1 + 4\tau^2 f^2}}(2\tau f\mathbf{e} - \mathbf{n}).$$

Then it holds that

$$(2.14) \quad \dot{\gamma} = \frac{2\tau}{\sqrt{1 + 4\tau^2 f^2}}(\mathbf{u}_1 - 2\tau f\mathbf{u}_2).$$

Applying (2.13), we have that

$$(2.15) \quad \frac{d}{d\tau}(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1, \mathbf{u}_2) \frac{2(2f + 4\tau^2 f^3 + \tau\dot{f})}{1 + 4\tau^2 f^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using this observation, we shall now prove the last assertion. We fix a  $C^\infty$ -function  $f \in C_0^\infty(\mathbf{R})$  satisfying  $f(0) \neq 0$ . Let  $\gamma(\tau)$  be a solution of the system of ordinary differential equations (2.14) and (2.15) with the initial conditions

$$\gamma(0) = \mathbf{0}, \quad \mathbf{u}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By (2.14), we have

$$(2.16) \quad |\dot{\gamma}| = 2|\tau|.$$

On the other hand, one can verify that

$$(2.17) \quad \ddot{\gamma} = 2\sqrt{1 + 4\tau^2 f^2}\mathbf{u}_1,$$

$$(2.18) \quad \gamma^{(3)}(0) = 8f(0)\mathbf{u}_2(0),$$

and then we have that

$$[\ddot{\gamma}(0), \gamma^{(3)}(0)] = 16f(0) \neq 0,$$

which implies that  $\gamma(\tau)$  has a 3/2-cusp at  $\tau = 0$ . Then (2.16) and Remark 2.2 yield that  $\tau$  is the half-arclength parameter. By (2.14) and (2.17), the identity (1.3) follows immediately.  $\square$

We denote by  $S_0^*(\mathbf{R}, M^2)$  the set of germs of  $C^\infty$ -curves  $\gamma(t)$  on  $M^2$  which give a 3/2-cusp at  $t = 0$ . Then the map

$$(2.19) \quad \mathcal{F}_g : S_0^*(\mathbf{R}, M^2) \ni \gamma(t) \mapsto \sqrt{|s_g(t)|}\kappa_g(t) \in C_0^\infty(\mathbf{R})$$

is defined, namely  $\mathcal{F}_g(\gamma)$  is the normalized curvature function of the 3/2-cusp  $\gamma$ .

**Corollary 2.8.** *The image of the map  $\mathcal{F}_g$  coincides with the subset*

$$\Sigma_g := \{f \in C_0^\infty(\mathbf{R}); f(0) \neq 0\}.$$

*Proof.* Obviously the image of  $\mathcal{F}_g$  is contained in  $\Sigma_g$ . By applying Theorem 2.6, for each  $f \in \Sigma_g$ , there exist  $\delta > 0$  and a 3/2-cusp  $\gamma : (-\delta, \delta) \rightarrow (M^2, g)$  such that  $\sqrt{|s_g(t)|}\kappa_g(t) = f(t)$  and  $t$  is the half-arclength parameter. Namely,  $\mathcal{F}_g(\gamma) = f$  holds.  $\square$

### 3. AFFINE GEOMETRY OF CUSPS

Let  $\gamma(t)$  be a curve in  $\mathbf{R}^2$  defined on an interval  $(-\delta, \delta)$  for  $\delta > 0$ . We suppose that  $t = 0$  is a 3/2-cusp. The formula (1.2) suggests how to construct new invariants on singular points, and we shall now introduce an affine invariant of 3/2-cusps in the same manner. In this section, we assume that  $\gamma(t)$  has no inflection points for  $t \in (-\delta, \delta)$ . It is classically known that the affine curvature function is defined by (cf. [1] and [7])

$$(3.1) \quad \kappa_A = \frac{3[\dot{\gamma}, \ddot{\gamma}][\dot{\gamma}, \gamma^{(4)}] + 12[\dot{\gamma}, \ddot{\gamma}][\ddot{\gamma}, \gamma^{(3)}] - 5[\dot{\gamma}, \gamma^{(3)}]^2}{9[\dot{\gamma}, \ddot{\gamma}]^{8/3}},$$

which is invariant under equi-affine transformations and is independent of the choice of a parametrization  $t$  and of the orientation of  $\gamma$ . Here  $[\dot{\gamma}, \ddot{\gamma}]^{8/3}$  is positive because of our convention (1.5). The *affine arclength* of  $\gamma(t)$  is defined by

$$(3.2) \quad s_A(t) := \int_0^t |[\dot{\gamma}(u), \ddot{\gamma}(u)]^{1/3}| du,$$

which is invariant under equi-affine transformations in  $\mathbf{R}^2$ . This parameter satisfies

$$\varepsilon_A := [\dot{\gamma}(s_A), \ddot{\gamma}(s_A)] = \pm 1,$$

and

$$\kappa_A(s_A) = \varepsilon_A [\ddot{\gamma}(s_A), \gamma^{(3)}(s_A)].$$

It can be easily checked that  $\kappa_A$  diverges at 3/2-cusps. To prove Theorem 1.2, it is sufficient to show the following assertion, since orientation reversing equi-affine transformation of  $\mathbf{R}^2$  preserve  $\mu_A$  but reverse the signature of 3/2-cusps:

**Proposition 3.1.** *Let  $\gamma(t)$  be a positive 3/2-cusp at  $t = 0$  in the affine plane  $\mathbf{R}^2$ . Then there exist an orientation preserving equi-affine transformation  $T$  and a coordinate change  $t = t(u)$  such that  $dt/du > 0$  and*

$$(3.3) \quad T \circ \gamma \circ t(u) = \begin{pmatrix} u^2 \\ u^3 + \frac{\mu_A u^5}{80\sqrt[5]{54}} \end{pmatrix} + o(u^5),$$

where  $o(u^5)$  is a term of order higher than  $u^5$ .

*Proof.* We may assume that  $[\ddot{\gamma}(0), \gamma^{(3)}(0)] = 1$  by a suitable homothetic change of  $t$ . By a parallel translation of  $\mathbf{R}^2$ , we may set  $\gamma(0) = \mathbf{0}$ . If we set

$$T := (\ddot{\gamma}(0), \gamma^{(3)}(0)) \in \mathrm{SL}(2, \mathbf{R}),$$

then the new curve  $\Gamma_0(t) := T^{-1}\gamma(t)$  satisfies

$$\Gamma_0(0) = \dot{\Gamma}_0(0) = \mathbf{0}, \quad \ddot{\Gamma}_0(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Gamma_0^{(3)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



So we can write

$$\Gamma_0(t) = \begin{pmatrix} t^2 a_0(t)^2/2 \\ t^3 b_0(t)/6 \end{pmatrix},$$

where  $a_0(t)$  and  $b_0(t)$  are smooth functions at  $t = 0$  satisfying  $a_0(0) = b_0(0) = 1$ . If we set  $s := ta_0(t)$ , then it gives a new parametrization of  $\Gamma_0$  such that

$$\Gamma_0(v) = \begin{pmatrix} v^2/2 \\ v^3 b(v)/6 \end{pmatrix},$$

where  $b(v)$  is a smooth function at  $v = 0$  satisfying  $b(0) = 1$ . Next, we set  $v = 12^{1/5}w$  and

$$\Gamma(w) := \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix} \Gamma_0(w) \quad (k := \frac{6}{12^{3/5}}).$$

Then we can write

$$\Gamma(u) = \begin{pmatrix} w^2 \\ w^3(1 + wB(w)) \end{pmatrix},$$

where  $B(w)$  is a smooth function at  $w = 0$ . Next, we set

$$\hat{\Gamma}(w) := \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \Gamma(w) = \begin{pmatrix} w^2(1 + \xi w(1 + wB(w))) \\ w^3(1 + wB(w)) \end{pmatrix},$$

where  $\xi$  is a constant. If we take a new parameter  $u := w\sqrt{1 + \xi w + w^2 \xi B(w)}$ , then it satisfies

$$u^3 = w^3 + \frac{3\xi w^4}{2} + o(w^4).$$

So if we set  $\xi := 2B(0)/3$ , it holds that

$$\hat{\Gamma}(u) = \begin{pmatrix} u^2 \\ u^3 + cu^5 + o(u^5) \end{pmatrix}.$$

By (1.4), we have  $c = \mu_A/(80\sqrt[5]{54})$ . □

To prove Theorem 1.3, we prepare the following assertion:

**Proposition 3.2.** *Let  $t = 0$  be a 3/2-cusp of a curve  $\gamma(t)$  in the affine plane  $\mathbf{R}^2$ . Then  $\tau := (s_A)^{3/5}$  can be taken as a new local parametrization of  $\gamma(t)$  at  $t = 0$ . (We shall call  $\tau$  the 3/5-arclength parameter.)*

*Proof.* Without loss of generality, we may assume that  $\gamma(0) = \mathbf{0}$ . Then we have the expression  $\gamma(t) = t^2 \Gamma(t)$ , where  $\Gamma(t)$  is a smooth  $\mathbf{R}^2$ -valued function around  $t = 0$ . Using this, we can write  $[\dot{\gamma}, \dot{\gamma}] = t^2 a(t)$ , where  $a \in C_0^\infty(\mathbf{R})$  satisfies  $a(0) \neq 0$ . Now applying Lemma .9 by setting  $\alpha = 2/3$  and  $\varphi(t) = a(t)$ , the function  $f(t) = t^{-5/3} s_A$  is a  $C^\infty$ -function satisfying  $f(0) \neq 0$ . Thus

$$\tau := (s_A)^{3/5} = t f(t)^{3/5}$$

gives a new local parametrization of the curve  $\gamma$ . □

*Remark 3.3.* Let  $\gamma(\tau)$  be a 3/2-cusp at  $\tau = 0$ . If  $\tau$  is the 3/5-arclength parameter, it holds that

$$(3.4) \quad \dot{\gamma} = \frac{5}{3} \tau^{2/3} \gamma', \quad \ddot{\gamma} = \frac{5(2\gamma' + 5\tau^{5/3} \gamma'')}{9\tau^{1/3}},$$

where we use the notations as in (2.10) and (2.11), namely, the dot (resp. the prime) means the derivative with respect to the 3/5-arclength parameter  $\tau$  (resp.

the affine arclength parameter  $s_A$ ). Since  $[\gamma', \gamma''] = \pm 1$ , it can be easily seen that the parameter  $t$  is the  $3/5$ -arclength parameter if and only if  $[\dot{\gamma}(t), \ddot{\gamma}(t)] = \pm 125t^2/27$ .

**Corollary 3.4.** *Let  $t = 0$  be a  $3/2$ -cusp of a curve  $\gamma(t)$  in  $\mathbf{R}^2$ . Then  $(s_A)^2 \kappa_A$  is a  $C^\infty$ -function at  $t = 0$ . (As mentioned in the introduction, we call  $(s_A)^2 \kappa_A$  the normalized affine curvature function at the cusp  $t = 0$ .)*

*Proof.* Without loss of generality, we may assume that  $t = 0$  is a positive cusp. Then it holds that  $[\dot{\gamma}, \ddot{\gamma}] > 0$ . We may assume that  $\gamma(0) = \mathbf{0}$ . Then we may set

$$(3.5) \quad \gamma(t) = \frac{\ddot{\gamma}(0)}{2!}t^2 + \frac{\gamma^{(3)}(0)}{3!}t^3 + t^4\Gamma(t),$$

where  $\Gamma(t)$  is a smooth  $\mathbf{R}^2$ -valued function around  $t = 0$ . For the sake of simplicity, we set  $d_{23} := [\ddot{\gamma}(0), \gamma^{(3)}(0)] (\neq 0)$ . Then (3.5) yields the following expression

$$[\dot{\gamma}, \ddot{\gamma}] = t^2 a_1(t), \quad [\dot{\gamma}, \gamma^{(3)}] = t a_2(t), \quad [\ddot{\gamma}, \gamma^{(3)}] = a_3(t), \quad [\dot{\gamma}, \gamma^{(4)}] = t a_4(t),$$

where  $a_j(t)$  ( $j = 1, 2, 3, 4$ ) are smooth functions of  $t$ . In particular,  $\kappa_A$  as in (3.1) satisfies  $\kappa_A = t^{-10/3}\psi_1(t)$ , where  $\psi_1(t) \in C_0^\infty(\mathbf{R})$ . Since  $s_A$  can be expressed by  $s_A = t^{5/3}\psi_2(t)$  ( $\psi_2 \in C_0^\infty(\mathbf{R})$ ) (cf. Proposition 3.2), we get the assertion.  $\square$

*Proof of Theorem 1.3.* Let  $\gamma(\tau)$  be a  $3/2$ -cusp at  $\tau = 0$  and  $\tau$  the  $3/5$ -arclength parameter. Differentiating (3.4), we have the following identity

$$(3.6) \quad \gamma^{(3)} = \frac{5(-2\gamma' + 30\gamma''\tau^{5/3} + 25\tau^{10/3}\gamma^{[3]})}{27\tau^{4/3}},$$

where we use the notations as in (2.10) and (2.11). By Corollary 3.4, we may set  $(s_A)^2 \kappa_A = f(\tau)$ . Then substituting the identity  $\gamma^{[3]} = -\tau^{-10/3}f\gamma'$  into (3.6), we have

$$(3.7) \quad \gamma^{(3)} = \frac{5}{27\tau^{4/3}} \left( -(2 + 25f)\gamma' + 30\tau^{5/3}\gamma'' \right).$$

Differentiating (3.7) by using  $\gamma^{[3]} = -\tau^{-10/3}f\gamma'$  again, we have

$$(3.8) \quad \gamma^{(4)} = -\frac{5}{81\tau^{7/3}} \left( (75\tau f + 50f - 8)\gamma' + 5\tau^{5/3}(25f - 4)\gamma'' \right).$$

Since  $[\gamma', \gamma''] = 1$ , it holds that

$$(3.9) \quad [\dot{\gamma}(\tau), \gamma^{(4)}(\tau)] = \frac{5^3}{3^5} \left( 4 - 25f(\tau) \right).$$

Since  $\dot{\gamma}(0) = \mathbf{0}$ , we have  $f(0) = 4/25$ . So there exists a function  $g \in C_0^\infty(\mathbf{R})$  such that  $f = (4/25) + \tau g$ . Then it holds that

$$(3.10) \quad [\gamma^{(3)}, \gamma^{(4)}] = \frac{5^5 (18\tau g + 25\tau g^2 + 36g)}{3^7 \tau}.$$

Since the left hand side is smooth at  $\tau = 0$ , we can conclude that  $g(0) = 0$ . So we may set  $f = (4/25) + \tau^2 h$ , where  $h \in C_0^\infty(\mathbf{R})$ . Using this expression and the

relation  $[\gamma', \gamma''] = 1$ , we have that

$$(3.11) \quad [\dot{\gamma}, \ddot{\gamma}] = \frac{125\tau^2}{27}, \quad [\dot{\gamma}, \gamma^{(3)}] = \frac{250\tau}{27}, \quad [\ddot{\gamma}, \gamma^{(3)}] = \frac{5^3}{3^5} (18 + 25\tau^2 h),$$

$$(3.12) \quad [\dot{\gamma}, \gamma^{(4)}] = -\frac{5^5}{3^5} \tau^2 h(t), \quad [\ddot{\gamma}, \gamma^{(4)}] = \frac{5^5}{3^5} \tau (\tau \dot{h} + 2h),$$

$$(3.13) \quad [\gamma^{(3)}, \gamma^{(4)}] = \frac{5^5}{3^7} (18\tau \dot{h} + 25\tau^2 h^2 + 54h).$$

Since

$$\frac{5^5}{3^5} \tau (3\dot{h} + \tau \ddot{h}) = \frac{d}{d\tau} [\ddot{\gamma}, \gamma^{(4)}] = [\gamma^{(3)}, \gamma^{(4)}] + [\ddot{\gamma}, \gamma^{(5)}],$$

we have  $[\ddot{\gamma}(0), \gamma^{(5)}(0)] = -4h(0)(5/3)^5$ . Using these relations and (1.4), one can easily check that

$$\frac{1}{220} \sqrt[5]{\frac{20}{3}} \mu_A = h(0) \left( = \lim_{\tau \rightarrow 0} \frac{(s_A)^2 \kappa_A - 4/25}{\tau^2} \right).$$

Since both sides of (1.6) are independent of the choice of parameters of the curve, this proves the identity (1.6).

Next we prove the last part of the theorem. By (3.11),  $\gamma(\tau)$  must satisfy the initial conditions

$$(3.14) \quad \dot{\gamma}(0) = \mathbf{0}, \quad [\ddot{\gamma}(0), \gamma^{(3)}(0)] = \frac{250}{27}.$$

Since  $\ddot{\gamma}(0), \gamma^{(3)}(0)$  are linearly independent, we can write

$$(3.15) \quad \dot{\gamma} = \alpha_1 \ddot{\gamma} + \alpha_2 \gamma^{(3)}, \quad \gamma^{(4)} = \beta_1 \ddot{\gamma} + \beta_2 \gamma^{(3)},$$

where

$$(3.16) \quad \begin{aligned} \alpha_1 &= \frac{[\dot{\gamma}, \gamma^{(3)}]}{[\ddot{\gamma}, \gamma^{(3)}]} = \frac{18\tau}{18 + 25\tau^2 h}, \quad \alpha_2 = -\frac{[\dot{\gamma}, \ddot{\gamma}]}{[\ddot{\gamma}, \gamma^{(3)}]} = -\frac{9\tau^2}{18 + 25\tau^2 h}, \\ \beta_1 &= -\frac{[\gamma^{(3)}, \gamma^{(4)}]}{[\ddot{\gamma}, \gamma^{(3)}]} = -\frac{25(18\tau \dot{h} + 25\tau^2 h^2 + 54h)}{9(18 + 25\tau^2 h)}, \\ \beta_2 &= -\frac{[\ddot{\gamma}, \gamma^{(4)}]}{[\ddot{\gamma}, \gamma^{(3)}]} = \frac{25\tau(\tau \dot{h} + 2h)}{18 + 25\tau^2 h}. \end{aligned}$$

In particular,  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C_0^\infty(\mathbf{R})$ . We now fix  $h \in C_0^\infty(\mathbf{R})$ , and take a solution  $\gamma(\tau)$  of the ordinary differential equation

$$(3.17) \quad \dot{\gamma} = \alpha_1 \xi + \alpha_2 \eta, \quad \dot{\xi} = \eta, \quad \dot{\eta} = \beta_1 \xi + \beta_2 \eta$$

with the initial conditions

$$(3.18) \quad \gamma(0) = \dot{\gamma}(0) = \mathbf{0}, \quad \xi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta(0) = \frac{250}{27} \mathbf{e}_2.$$

By (3.16), it holds that

$$\frac{d}{d\tau} \log |[\ddot{\gamma}, \gamma^{(3)}]| = \frac{[\ddot{\gamma}, \gamma^{(4)}]}{[\ddot{\gamma}, \gamma^{(3)}]} = \beta_2 = \frac{d}{d\tau} \log |18 + 25\tau^2 h|,$$

and there exists a positive constant  $C$  such that

$$[\ddot{\gamma}, \gamma^{(3)}] = C(18 + 25\tau^2 h).$$

Then (3.18) yields that  $C = 125/243$ , and we get

$$(3.19) \quad [\dot{\gamma}, \ddot{\gamma}] = 125\tau^2/27,$$

which implies that  $\tau = 0$  is a  $3/2$ -cusp and  $\tau$  is the  $3/5$ -arclength parameter of  $\gamma$ . Moreover, (3.19) yields that

$$[\dot{\gamma}, \gamma^{(3)}] = \frac{d}{d\tau}[\dot{\gamma}, \ddot{\gamma}] = \frac{250\tau}{27}$$

and

$$[\dot{\gamma}, \gamma^{(4)}] = \beta_1[\dot{\gamma}, \ddot{\gamma}] + \beta_2[\dot{\gamma}, \gamma^{(3)}] = -\left(\frac{5}{3}\right)^5 \tau^2 h.$$

Using these identities, one can easily check that  $(s_A)^2 \kappa_A$  is equal to  $(4/25) + \tau^2 h$ .  $\square$

*Example 3.5.* The normalized affine curvature function of the cusp  $\gamma_0(t) = {}^t(t^2, t^3)$  is identically equal to  $4/25$ . The parameter  $t$  is proportional to the  $3/5$ -arclength parameter. It is interesting to consider the family of curves  $\sigma_c(\tau)$  with  $3/5$ -arclength parameter  $\tau$  whose normalized affine curvature function is equal to  $(4/25) + c\tau^2$ . When  $c = 0$ , then  $\sigma_0$  is equal to the cusp  $\gamma_0$ . By solving the ordinary differential equation (3.15) with the initial data (3.18), we get the figure of  $\sigma_c(\tau)$  for  $c = \pm 1$  (see Figure 2).

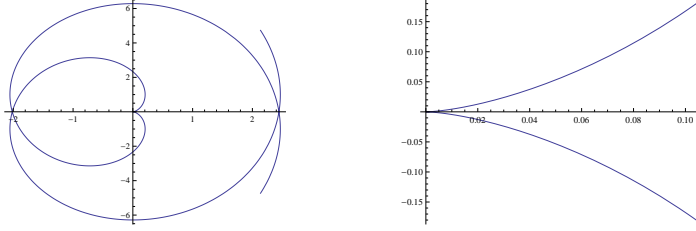


FIGURE 2. The  $3/2$ -cusps  $\sigma_1$  (left) and  $\sigma_{-1}$  (right)

*Example 3.6.* The cycloid (2.7) has a positive cusp  $t = 0$  with positive affine cuspidal curvature  $\mu_A = 36a^{-4/5}$ . (The sign of the affine cuspidal curvature is independent of the signature of cusps.)

*Example 3.7.* The hyperbolic cycloid

$$\gamma(t) = a \begin{pmatrix} t - \sinh t \\ -1 + \cosh t \end{pmatrix} \quad (a > 0)$$

has a positive cusp  $t = 0$  with negative affine cuspidal curvature  $\mu_A = -36a^{-4/5}$ .

To finish this section, we generalize Theorem 1.3 for cusps in an arbitrary 2-manifold with an equi-affine structure (cf. [3]): Let  $M^2$  be an oriented 2-manifold with an affine connection  $D$  (which may not be torsion free). If there exists a non-vanishing parallel positively oriented 2-form  $\Omega$  defined on  $M^2$  with respect to  $D$ , then the triple  $(M^2, D, \Omega)$  is called an *equi-affine 2-manifold*. (In the usual definition of equi-affine structure,  $D$  is a torsion free connection having symmetric Ricci tensor. However, we do not assume this here, since it is not needed for the following discussions.) For the sake of simplicity, we set

$$[v, w] := \Omega(v, w) \quad (v, w \in T_p M^2, p \in M^2).$$

Let  $\gamma(t)$  ( $|t| < \delta$ ) be a regular curve in  $M^2$ . Then the affine curvature function is defined by exactly the same formula (3.1), where

$$\ddot{\gamma}(t) = D_t \dot{\gamma}(t), \quad \gamma^{(3)} = D_t \ddot{\gamma}, \quad \gamma^{(4)} = D_t \gamma^{(3)}.$$

If  $(M^2, g)$  is an orientable Riemannian 2-manifold, then the area element  $\Omega_g$  is globally defined on  $M^2$  which is a parallel 2-form with respect to the Levi-Civita connection. Then  $(M^2, \nabla, \Omega_g)$  gives a typical example of an equi-affine 2-manifold.

Let  $\gamma(t)$  ( $|t| < \delta$ ) be a 3/2-cusp at  $t = 0$  on  $M^2$ . We define the *affine cuspidal curvature*  $\mu_A$  at  $t = 0$  by (1.4), as a generalization of the case of  $\mathbf{R}^2$ . Then the following assertion holds, which is a generalization of Theorem 1.3.

**Theorem 3.8.** *Let  $\gamma(t)$  ( $|t| < \delta$ ) be a 3/2-cusp at  $t = 0$  in an equi-affine 2-manifold  $(M^2, D, \Omega)$ . Then the same assertion as in Theorem 1.3 holds.*

By Lemma 2.7, we have the expression

$$\dot{\gamma}(t) = P_2 t + \frac{P_3}{2} t^2 + t^3 \Gamma(t),$$

where  $P_j := P_j(t)$  ( $j = 1, 2, 3, \dots$ ) is a parallel vector field along  $\gamma(t)$  such that  $P_j(0) = \gamma^{(j)}(0) \in T_{\gamma(0)} M^2$ , and  $\Gamma(t)$  is a  $C^\infty$ -vector field along  $\gamma$ . Using this expression, like as in the proof of Theorem 2.6, we can prove Corollary 3.4. The remainder of the proof is exactly the same as that of Theorem 1.3. Like as in the Euclidean case (cf. (3.20)), the map

$$(3.20) \quad \mathcal{F}_A : S_0^*(\mathbf{R}, M^2) \ni \gamma(t) \mapsto s_A(t)^2 \kappa_A(t) \in C_0^\infty(\mathbf{R})$$

is defined, namely  $\mathcal{F}_A(\gamma)$  is the normalized affine curvature function of the 3/2-cusp  $\gamma$  with respect to the equi-affine structure  $(D, \Omega)$ . The following assertion follows immediately.

**Corollary 3.9.** *The image of the map  $\mathcal{F}_A$  coincides with the subset*

$$\Sigma_A := \left\{ f \in C_0^\infty(\mathbf{R}); f(0) = \frac{4}{25}, \dot{f}(0) = 0 \right\}.$$

#### 4. AFFINE GEOMETRY OF GENERIC INFLECTION POINTS

A point  $t = c$  of a regular curve  $\gamma(t)$  in the affine plane  $\mathbf{R}^2$  is called an *inflection point* if  $[\dot{\gamma}(c), \ddot{\gamma}(c)]$  vanishes. It is well-known that there is a duality between cusps and inflection points (cf. [5]). An inflection point  $t = c$  is called *generic* if  $[\dot{\gamma}(c), \gamma^{(3)}(c)]$  does not vanish. Fabricius-Bjerre [1] pointed out that  $\kappa_A$  diverges to  $-\infty$  for generic inflection points. More generally,  $\lim_{c \rightarrow 0} \inf_{|t| < c} \kappa_A(t)$  diverges to  $-\infty$  for arbitrary inflection points (see [7]).

**Definition 4.1.** A generic inflection point  $t = c$  is called *positive* (resp. *negative*) if  $[\dot{\gamma}(c), \gamma^{(3)}(c)]$  is positive (resp. negative).

This signature of inflection points is invariant under the orientation preserving diffeomorphisms of  $\mathbf{R}^2$  and is also invariant under the choice of an orientation of curves. In this section, we define a new affine invariant called the *affine inflectional curvature* at an inflection point  $t = c$  by

$$(4.1) \quad \mu_I := \varepsilon_I \frac{[\dot{\gamma}(c), \gamma^{(4)}(c)] - 6[\ddot{\gamma}(c), \gamma^{(3)}(c)]}{[\dot{\gamma}(c), \gamma^{(3)}(c)]^{5/4}},$$

where  $\varepsilon_I = \text{sgn}[\dot{\gamma}(c), \ddot{\gamma}(c)]$  is the signature of the inflection point. This is invariant under equi-affine transformations, but changes sign if we reverse the orientation of the curve  $\gamma$ . As an analogue of Theorem 1.2, we get the following assertion:

**Theorem 4.2.** *Let  $\gamma_1(t)$  and  $\gamma_2(t)$  be two positive generic inflection points at  $t = 0$  in the affine plane  $\mathbf{R}^2$ . Then  $\gamma_1$  and  $\gamma_2$  have the same inflectional curvature if and only if there exists an orientation preserving equi-affine transformation  $T$  of  $\mathbf{R}^2$  and a parametrization  $u = u(t)$  near  $t = 0$  such that  $du/dt > 0$  and  $|T \circ \gamma_2 \circ u(t) - \gamma_1(t)|$  has order higher than  $t^4$ .*

This assertion immediately follows from the following proposition.

**Proposition 4.3.** *Let  $\gamma(t)$  be a regular curve in  $\mathbf{R}^2$  having a positive generic inflection point at  $t = 0$ . Then there exist an orientation preserving equi-affine transformation  $T$  and a coordinate change  $t = t(u)$  such that*

$$(4.2) \quad T \circ \gamma \circ t(u) = \left( u^3 + \sqrt[4]{6} \mu_I u^4 / 4 \right) + o(u^4),$$

where  $o(u^4)$  is a term of order higher than  $u^4$ .

Since the proof of the proposition is similar to that of Proposition 4.3, we omit it here. We also get the following assertion:

**Theorem 4.4.** *Let  $t = 0$  be a generic inflection point of a regular curve  $\gamma(t)$  in  $\mathbf{R}^2$ . Then  $\tau = \text{sgn}(t)(s_A)^{3/4}$  can be taken to be a coordinate of  $\gamma$  at  $t = 0$  (called the 3/4-arclength parameter) and  $f := (s_A)^2 \kappa_A$  is a  $C^\infty$ -function of  $t$  (and  $\tau$ ). Moreover, it satisfies (1.7), and*

$$(4.3) \quad \lim_{t \rightarrow 0} (s_A)^2 \kappa_A (= f(0)) = -\frac{5}{16},$$

$$(4.4) \quad \lim_{t \rightarrow 0} \frac{(s_A)^2 \kappa_A + 5/16}{\tau} = -\frac{3\sqrt[4]{3}}{28\sqrt{2}} \mu_I.$$

Furthermore, if  $t = \tau$ , then (1.7) is reduced to the relation

$$(4.5) \quad 32 \left( \frac{df(0)}{d\tau} \right)^2 + 9 \frac{d^2 f(0)}{d\tau^2} = 0.$$

Conversely, if we take a  $C^\infty$ -function  $f(\tau)$  satisfying  $f(0) = -5/16$  and (4.5), then there exists a generic inflection point whose normalized affine curvature function is  $f(\tau)$  with respect to the 3/4-arclength parameter.

*Proof.* Let  $t = 0$  be a generic inflection point of a plane curve  $\gamma(t)$ . Since orientation reversing equi-affine transformations preserve  $\kappa_A$  and  $\mu_I$ , we may assume that  $t = 0$  is a positive inflection point. As an analogue of Proposition 3.2,  $\tau = \text{sgn}(t)(s_A)^{3/4}$  can be taken to be a coordinate of  $\gamma$  at  $t = 0$ . (As in Theorem 4.4,  $\tau$  is called the 3/4-arclength parameter.) Also, we can prove that  $f(t) := (s_A)^2 \kappa_A$  is a  $C^\infty$ -function of  $t$ .

The following identities hold (cf. (1.5))

$$(4.6) \quad \dot{\gamma} = \frac{4}{3} \tau^{1/3} \gamma', \quad \ddot{\gamma} = \frac{4(\gamma' + 4\tau^{4/3} \gamma'')}{9\tau^{2/3}},$$

where we use the notations as in (2.10) and (2.11), namely, the dot (resp. the prime) means the derivative with respect to the 3/4-arclength parameter  $\tau$  (resp.

the affine arclength parameter  $s_A$ ). Using (4.6), one can easily check that  $t$  is a  $3/4$ -arclength parameter (of a positive inflection point) if and only if  $[\dot{\gamma}, \ddot{\gamma}] = 64t/27$ . By (4.6), it holds that

$$\gamma^{(3)} = \frac{8(-\gamma' + 6\tau^{4/3}\gamma'' + 8\tau^{8/3}\gamma^{[3]})}{27\tau^{5/3}}.$$

Substituting  $\gamma^{[3]} = -\tau^{-8/3}f\gamma'$ , we get

$$\gamma^{(3)} = -\frac{8((8f+1)\gamma' - 6\tau^{4/3}\gamma'')}{27\tau^{5/3}}.$$

Differentiating it by using  $\gamma^{[3]} = -\tau^{-8/3}f\gamma'$  again, we have

$$\gamma^{(4)} = \frac{8}{81\tau^{8/3}} \left( (-24\tau f' + 16f + 5)\gamma' - 2\tau^{4/3}(16f + 5)\gamma'' \right).$$

Since  $[\gamma', \gamma''] = 1$ , it holds that

$$(4.7) \quad [\dot{\gamma}, \ddot{\gamma}] = \frac{64}{27}\tau, \quad [\dot{\gamma}, \gamma^{(3)}] = \frac{64}{27}, \quad [\ddot{\gamma}, \gamma^{(3)}] = \frac{64(5 + 16c + 16\tau g(\tau))}{243\tau},$$

where we set  $(s_A)^2\kappa_A = c + \tau g(\tau)$  ( $g \in C_0^\infty(\mathbf{R})$ ). Since  $[\ddot{\gamma}, \gamma^{(3)}] \in C_0^\infty(\mathbf{R})$ , we have that  $c = -5/16$ . Moreover, since  $[\dot{\gamma}, \gamma^{(3)}]$  is a constant function,  $[\dot{\gamma}, \gamma^{(4)}]$  and  $-\ddot{\gamma}, \gamma^{(3)}$  are both equal to  $-2^{10}g(\tau)/3^5$ . Using these relations and (4.1), one can easily check that

$$-\frac{3\sqrt[4]{3}}{28\sqrt{2}}\mu_A = g(0) \left( = \lim_{\tau \rightarrow 0} \frac{(s_A)^2\kappa_A + 5/16}{\tau} \right),$$

which proves the identity (4.4). On the other hand, it holds that

$$[\gamma^{(3)}, \gamma^{(4)}] = \frac{2^{10}(9\dot{g}(\tau) + 16g(\tau)^2)}{3^7\tau}.$$

Since  $[\gamma^{(3)}, \gamma^{(4)}] \in C_0^\infty(\mathbf{R})$ , we can conclude  $9\dot{g}(0) + 16g(0)^2 = 0$ , which is equivalent to the condition (4.5). In this situation, there exists a smooth function  $h(\tau)$  near  $\tau = 0$  such that

$$(4.8) \quad 9\dot{g}(\tau) + 16g(\tau)^2 = \tau h(\tau).$$

Since  $\dot{\gamma}$  and  $\gamma^{(3)}$  are linearly independent, we can write

$$(4.9) \quad \ddot{\gamma} = a_{11}\dot{\gamma} + a_{12}\gamma^{(3)}, \quad \gamma^{(4)} = a_{21}\dot{\gamma} + a_{22}\gamma^{(3)}.$$

By (4.8), we have

$$(4.10) \quad a_{11} = \frac{[\ddot{\gamma}, \gamma^{(3)}]}{[\dot{\gamma}, \gamma^{(3)}]} = \frac{16g(\tau)}{9}, \quad a_{12} = \frac{[\dot{\gamma}, \ddot{\gamma}]}{[\dot{\gamma}, \gamma^{(3)}]} = \tau,$$

$$(4.11) \quad a_{21} = -\frac{[\gamma^{(3)}, \gamma^{(4)}]}{[\dot{\gamma}, \gamma^{(3)}]} = -\frac{16h(\tau)}{81}, \quad a_{22} = \frac{[\dot{\gamma}, \gamma^{(4)}]}{[\dot{\gamma}, \gamma^{(3)}]} = -\frac{16g(\tau)}{9}.$$

We now take a function  $f \in C_0^\infty(\mathbf{R})$  satisfying (4.5). Let  $g(\tau)$  be the function defined by  $g(\tau) = \tau f(\tau)$ , and  $h(\tau)$  be the function given by (4.8). We consider the ordinary differential equation

$$(4.12) \quad \dot{\gamma} = \xi, \quad \dot{\xi} = a_{11}\xi + a_{12}\eta, \quad \dot{\eta} = a_{21}\xi + a_{22}\eta,$$

where  $a_{ij}$  ( $i, j = 1, 2$ ) are defined by (4.10) and (4.11). Then there exists a solution  $\gamma(\tau)$  of the ordinary differential equation (4.12) satisfying the initial conditions

$$\gamma(0) = \mathbf{0}, \quad \xi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta(0) = \frac{64}{27}\mathbf{e}_2.$$

By (4.12) and (4.10), it can be easily checked that  $\gamma^{(3)} = \ddot{\xi} = \eta$ . Then  $\varphi := [\dot{\gamma}, \ddot{\gamma}]$  satisfies

$$\varphi = [\xi, \dot{\xi}] = a_{12}[\xi, \eta] = a_{12}[\dot{\gamma}, \gamma^{(3)}] = a_{12}\dot{\varphi},$$

which yields that

$$\frac{d}{d\tau} \log |\varphi| = \frac{1}{a_{12}} = \frac{1}{\tau} = \frac{d}{d\tau} \log |\tau|.$$

Thus there exists a positive constant  $C$  such that  $\varphi = C\tau$ . Since

$$\dot{\varphi}(0) = [\dot{\gamma}(0), \gamma^{(3)}(0)] = [\xi(0), \eta(0)] = \frac{64}{27},$$

we have  $C = 64/27$ . Then

$$(4.13) \quad [\dot{\gamma}(\tau), \gamma^{(3)}(\tau)] = \frac{64}{27},$$

which implies that  $\tau = 0$  is a generic inflection point and  $\tau$  is the 3/4-arclength parameter. Moreover, we can get the relation  $[\dot{\gamma}, \gamma^{(4)}] = -2^{10}g(\tau)/3^5$  from the identity  $\gamma^{(4)} = a_{21}\dot{\gamma} + a_{22}\gamma^{(3)}$ . Since  $[\dot{\gamma}, \gamma^{(3)}]$  is a constant,

$$(4.14) \quad [\ddot{\gamma}, \gamma^{(3)}] = -[\dot{\gamma}, \gamma^{(4)}]$$

holds. Using these relations, it can be easily checked that  $(s_A)^2\kappa_A$  is equal to  $-(5/16) + \tau g(\tau)$ , which proves last assertion of Theorem 4.4.

Finally, using the initial parameter  $t$  of  $\gamma(t)$ , we have that

$$32(\dot{f})^2 + 9\ddot{f} = 32\left(\frac{dt}{d\tau}\right)^2 f_t^2 + 9\frac{d^2t}{d\tau^2} f_t + 9\left(\frac{dt}{d\tau}\right)^2 f_{tt},$$

where  $f_t := df/dt$  and  $f_{tt} := d^2f/dt^2$ . Since

$$\frac{d^2t/d\tau^2}{(dt/d\tau)^2} = -\frac{d}{d\tau} \left( \frac{1}{dt/d\tau} \right) = -\frac{dt}{d\tau} \frac{d}{dt} \left( \frac{d\tau}{dt} \right) = -\frac{d^2\tau/dt^2}{d\tau/dt},$$

(1.7) is reduced to the relation

$$32f_t(0)^2 - 9\frac{d^2\tau(0)/dt^2}{d\tau(0)/dt} f_t(0) + 9f_{tt}(0) = 0.$$

Since  $\tau = (s_A)^{3/4}$ , one can prove the following identity using L'Hospital's rule

$$\begin{aligned} \frac{d\tau(0)}{dt} &= \left(\frac{3}{4}\right)^{3/4} [\gamma_t(0), \gamma_{ttt}(0)]^{1/4}, \\ \frac{d^2\tau(0)}{dt^2} &= \left(\frac{3}{4}\right)^{3/4} \frac{[\gamma_t(0), \gamma_{tttt}(0)] + [\gamma_{tt}(0), \gamma_{ttt}(0)]}{7[\gamma_t(0), \gamma_{ttt}(0)]^{3/4}}, \end{aligned}$$

which yield the identity (1.7), where  $\gamma_t := d\gamma/dt$  and  $\gamma_{tt} := d^2\gamma/dt^2$  etc.  $\square$

*Example 4.5.* The normalized affine curvature function of the inflection point  $\gamma(t) = {}^t(at, at^3)$  ( $a > 0$ ) is identically equal to  $-5/16$ . The parameter  $t$  is proportional to the 3/4-arclength parameter.



*Example 4.6.* The *skew-cycloid*

$$\gamma(t) = a(t - \sin t, -t + \cos t) \quad (a > 0)$$

has a positive inflection point at  $t = 0$  of inflectional curvature  $\mu_I = -6/\sqrt{a}$ . If one reverse the orientation,  $\gamma(-t)$  also has a positive inflection point with the inflectional curvature  $\mu_I = 6/\sqrt{a}$ .

Like as in the case of cusps, the following assertion can be proved by modifying the proof of Theorem 4.4:

**Theorem 4.7.** *Let  $\gamma(t)$  ( $|t| < \delta$ ) be a generic inflection point at  $t = 0$  in an equi-affine 2-manifold  $(M^2, D, \Omega)$ . Then the same assertion as in Theorem 4.4 holds.*

We denote by  $I_0^*(\mathbf{R}, M^2)$  the set of germs of  $C^\infty$ -maps  $\gamma(t)$  defined on an open interval containing  $t = 0$  into an equi-affine 2-manifold  $(M^2, D, \Omega)$  which gives a generic inflection point at  $t = 0$ . Then the map

$$(4.15) \quad \mathcal{F}_I : I_0^*(\mathbf{R}, M^2) \ni \gamma(t) \mapsto s_A(t)^2 \kappa_A(t) \in C_0^\infty(\mathbf{R})$$

is defined, namely  $\mathcal{F}_I(\gamma)$  is the normalized affine curvature function of the generic inflection point  $\gamma$ .

**Corollary 4.8.** *The image of the map  $\mathcal{F}_I$  coincides with the subset*

$$\begin{aligned} \Sigma_I := & \left\{ f \in C_0^\infty(\mathbf{R}) ; f(0) = -\frac{5}{16}, \dot{f}(0) \neq 0 \right\} \\ & \cup \left\{ f \in C_0^\infty(\mathbf{R}) ; f(0) = -\frac{5}{16}, \dot{f}(0) = \ddot{f}(0) = 0 \right\}. \end{aligned}$$

*Proof.* Obviously the image of  $\mathcal{F}_I$  is contained in  $\Sigma_I$ . We fix a function  $f \in C_0^\infty(\mathbf{R})$  such that  $f(0) = -5/16$ . If  $\dot{f}(0) = \ddot{f}(0) = 0$ , then  $f$  satisfies (4.5). On the other hand, if  $\dot{f}(0) \neq 0$ , then there exists a new parametrization  $\tau := t + ct^2$  such that  $f(\tau) := f \circ t(\tau)$  satisfies (4.5), by adjusting the constant  $c$ . Thus, for these two cases, Theorem 4.7 yields that there exist  $\delta > 0$  and a regular curve  $\gamma : (-\delta, \delta) \rightarrow (M^2, g)$  having an inflection point  $\tau = 0$  such that  $s_A(\tau)^2 \kappa_A(\tau) = f(\tau)$  and  $\tau$  is the 3/4-arclength parameter. Since the relation  $s_A(\tau)^2 \kappa_A(\tau) = f(\tau)$  is independent of the choice of parameters, we get the assertion.  $\square$

#### APPENDIX. A DIVISION LEMMA

**Lemma .9.** *Let  $\varphi(t)$  be a  $C^\infty$ -function at  $t = 0$ , and  $\alpha$  a positive real number. Then the function defined by*

$$f(t) := \frac{\Phi(t)}{\operatorname{sgn}(t)|t|^{1+\alpha}} \quad (\Phi(t) := \int_0^t |u|^\alpha \varphi(u) du)$$

*is a  $C^\infty$ -function at  $t = 0$ , namely  $f \in C_0^\infty(\mathbf{R})$ . Moreover, it holds that*

$$(A.1) \quad f(0) = \frac{\varphi(0)}{1+\alpha}.$$

*Proof.* In fact, we have that

$$\Phi(t) = \int_0^1 \frac{d\Phi(tu)}{du} du = \int_0^1 t \dot{\Phi}(tu) du = \operatorname{sgn}(t)|t|^{1+\alpha} \left( \int_0^1 |u|^\alpha \varphi(tu) du \right).$$

Since  $\alpha > 0$ , it follows that  $\int_0^1 |u|^\alpha \varphi(tu) du$  is a  $C^\infty$ -function at  $t = 0$ . By L'Hospital's rule, we have (A.1).  $\square$

If  $\alpha = 0$ , then  $f(t) = \Phi(t)/t$ , and the lemma reduces to the classical division lemma.

#### REFERENCES

- [1] Fr. Fabricius-Bjerre, *On a conjecture of Bol*, Math. Scand. 40 (1977), 194–196.
- [2] P. Giblin and G. Sapiro, *Affine-Invariant Distances, Envelopes and Symmetry Sets*, Geometriae Dedicata 71 (1998) 237–261.
- [3] S. Izumiya and T. Sano, *Generic affine differential geometry of plane curves*, Proc. Math. Soc. Edinburgh. 41 (1998) 315–324.
- [4] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, Cambridge, 1994.
- [5] K. Saji, M. Umehara and K. Yamada, *The duality between singular points and inflection points on wave fronts*, Osaka J. Math. 47 (2010), 591–607.
- [6] M. Umehara, *Differential geometry on surfaces with singularities* in The World of Singularities (ed. H. Arai, T. Sunada and K. Ueno) Nippon-Hyoron-sha Co., Ltd. (2005), 50–64, (Japanese).
- [7] M. Umehara, *A simplification of the proof of Bol’s conjecture on sextactic points*, Proc. Japan Acad. Ser. A 87 (2011), 10–12.

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